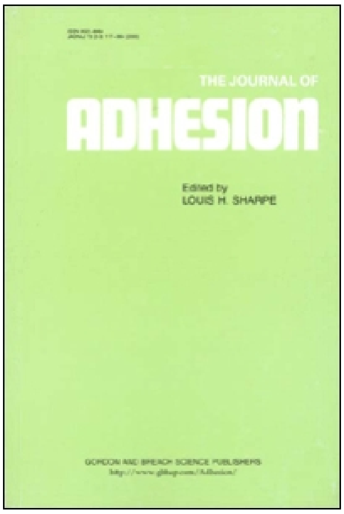


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# Two Finite Elements for Modeling the Adhesive in Bonded Configurations

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# Two Finite Elements for Modeling the Adhesive in Bonded Configurations†

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Two special finite elements are presented for modeling the adhesive in a bonded configuration. The assumptions of numerous lap joint theories can be modeled with the elements by selecting values of control parameters. An example is presented where control parameters are selected to model the assumptions of Delale and Erodogan and results with the elements are shown to converge to those of that reference. In another example, numerous combinations of control parameters are considered to study the effects of different assumptions on the maximum shear and normal stress in the adhesive.

KEY WORDS Bonded joints; lap joint assumptions; maximum adhesive stress; finite elements; adhesive element; offset nodes.

## INTRODUCTION

Goland and Reissner[1] presented in 1944 their classic paper on stresses in bonded joints. Other authors have extended their formulation to different configurations and/or constitutive relationships for the adherends.[2–9] References 10 and 11 give excellent reviews of research in this area. In general, these authors treat the adhesive in a similar fashion to that of Goland and Reissner. Other authors have attempted to make less restrictive assumptions about the adhesive as in Refs 12–14. Reference 12 allows stresses to vary through the thickness of the adhesive. Reference 13 is a general approach which removes many of the restrictions of earlier papers. The approach is, however, quite complex and has not yet been extended to configurations other than the single lap joint. Delale and Erdogan[14] also removed some of the restrictions and inconsistent assumptions found in the

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earlier theories. That reference is especially important in that it gives viscoelastic as well as elastic solutions. The viscoelastic solutions are analytical solutions to viscoelastic problems and thus provide check problems for numerical finite element based viscoelastic algorithms.

It might seem that a finite element analysis which idealizes both the adherends and adhesive with standard elements could be used to bypass the deficiencies of the lap joint theories and thus obtain the maximum stress in the adhesive. Such an approach has been tried in Refs 15–21. In most of these investigations, the investigators were not aware of the stress singularities which exist at the joint edges at the bi-material interfaces. The stress singularities make it impossible to obtain convergence of stress there due to the singular nature of the elasticity solution at those locations. References 22–25 have recognized the singular nature of the stress at joint edges at the bi-material interfaces and have calculated stress intensities at these locations. It has yet to be experimental verified whether stress intensities can be used as reliable strength parameters.

Recently Reddy and Roy[26,27] have performed non-linear analyses of adhesively bonded joints. In Ref. 26 they used an updated Lagrangian formulation to develop a 2-D finite element which accounts for geometric non-linearity. In Ref. 27, they examined viscoelasticity and diffusion in adhesively bonded joints and reported results using their special finite element program NOVA.

This paper is concerned with the linear problem only. It uses special adhesive elements to model the adhesive. The elements are derived using incomplete strain-displacement equations or incomplete strain displacement equations together with incomplete stress–strain equations to give a simplified state of stress in the adhesive. The stress singularities disappear with these elements and thus it is possible to obtain convergence of adhesive stress with mesh refinement. Maximum adhesive stresses thus obtained are, of course, limited by the fact that one or more of the equations of elasticity are being violated. The stress intensity approach, however, is also limited in that it is based on the linearized strain displacement equations. This linearization is obviously not valid when infinite strains are being predicted. One could obtain a finite element solution to the bonded connection problem using the nonlinear equations of elasticity. However, such analyses require sophisticated finite element programs which are not readily available and such analyses are computationally expensive to obtain. Of these choices, the best seems to be to use special adhesive elements which model reasonable lap joint theories. In this way, the power of the finite element method can be used to analyze complex geometries, and performance using the elements can be tested against corresponding analytical solutions.

Different adhesive elements can be developed, depending on the assumptions made in the element derivation. Carpenter[28] developed a 2-node element to model Goland and Reissner's[1] zero-thickness adhesive assumption (thickness assumptions are discussed in detail in a later section of this paper). Later, Carpenter[29] developed another 2-node element which could model Goland and Reissner's[1] finite thickness adhesive assumption as well as the assumptions of Ojalvo and Eidinoff.[12] References 11 and 30 develop a 6-node adhesive element

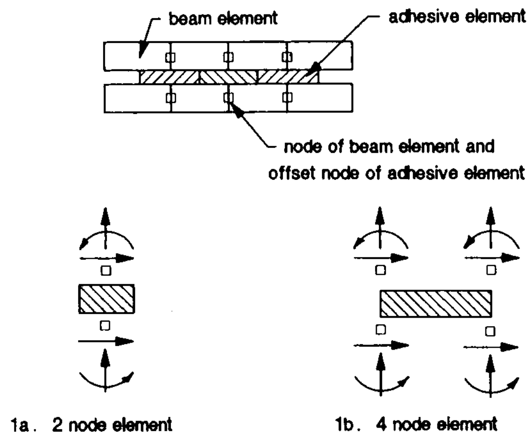

FIGURE 1 Finite element idealizations.



that models the assumptions of Delale and Erdogan.[14] Here the adherends were modeled with isoparametric elements as opposed to plate elements as in Ref. 14.

The present paper presents 2-node and 4-node adhesive elements that are general enough that they can model the assumptions of numerous authors. The type of assumptions to be made are set by various control parameters. Example 1 is presented where the elements are used to model the assumptions of Ref. 14 and the results are shown to converge to those of that reference. In Example 2, numerous combinations of control parameters are considered to study the effects of different assumptions on the maximum shear and normal stress in the adhesive.

## FINITE ELEMENT IDEALIZATIONS

Adherends can be modeled conveniently with standard beam-type elements. These elements, however, normally have their nodes along the centroid of the element. Thus, in this study, the adhesive is modeled with special elements with offset nodes which correspond to the nodes of the adherend elements. Figure 1a shows a finite element idealization of a bonded configuration where the adhesive element is a 2-node element with offset nodes. Figure 1b shows a 4-node adhesive element with offset nodes. Specifics concerning the adhesive and adherend elements are next discussed in some detail.

## FOUR NODE ADHESIVE ELEMENT

Figure 2 shows a 4-node adhesive element (the offset nodes are not shown). The displacements within the element are assumed to vary linearly in the $x$ and $z$

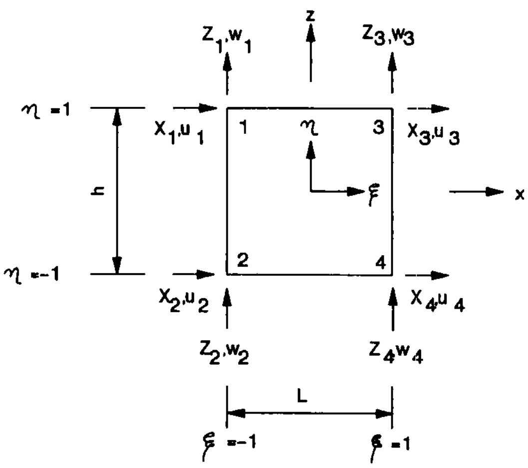


FIGURE 2 Four node adhesive element.



direction thus

$$u = \left[\frac{(u_1 + u_2)}{2} + \frac{z}{h}(u_1 - u_2)\right]\left[1 - \frac{x}{L}\right] + \left[\frac{(u_3 + u_4)}{2} + \frac{z}{h}(u_3 - u_4)\right]\left[\frac{x}{L}\right] \quad (1)$$

$$w = \left[\frac{(w_1 + w_2)}{2} + \frac{z}{h}(w_1 - w_2)\right]\left[1 - \frac{x}{L}\right] + \left[\frac{(w_3 + w_4)}{2} + \frac{z}{h}(w_3 - w_4)\right]\left[\frac{x}{L}\right] \quad (2)$$

which is the standard isoparametric displacement assumptions.[31]

### Strain-displacement equations

Strain within the element using the linearized strain-displacement equations is found from Eqs (1) and (2) thus (see Eq. (3) opposite) or

$$\{\varepsilon\} = [B]\{\delta\} \quad (4)$$

In Eq. (3), the complete linearized strain-displacement equations require that $\alpha_1 = \alpha_2 = 1$. Various investigators have made assumptions concerning the strain-displacement relationship for the adhesive. The sundry assumptions can be modeled by assigning values to $\alpha_1$ and $\alpha_2$.

For example, authors such as Goland and Reissner,[1] Ojalvo and Eidinoff,[12] and Delale and Erdogan[14] use for the adhesive an incomplete shear strain-displacement equation thus:

$$\gamma_{xz} = \frac{\partial u}{\partial z}$$

Thus to model such an assumption

$$\alpha_1 = 0$$

$$
\begin{Bmatrix} \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{h}\left(1-\frac{x}{L}\right) & 0 & -\frac{1}{h}\left(1-\frac{x}{L}\right) & 0 & \frac{1}{h}\frac{x}{L} & 0 & -\frac{1}{h}\frac{x}{L} \\ -\frac{1}{2L}-\frac{z\alpha_2}{hL} & 0 & -\frac{1}{2L}+\frac{z\alpha_2}{hL} & 0 & \frac{1}{2L}+\frac{z\alpha_2}{hL} & 0 & \frac{1}{2L}-\frac{z\alpha_2}{hL} & 0 \\ \frac{1}{h}\left(1-\frac{x}{L}\right) & \alpha_1\left(-\frac{1}{2L}-\frac{z\alpha_2}{hL}\right) & -\frac{1}{h}\left(1-\frac{x}{L}\right) & \alpha_1\left(-\frac{1}{2L}+\frac{z\alpha_2}{hL}\right) & \frac{1}{h}\frac{x}{L} & \alpha_1\left(\frac{1}{2L}+\frac{z\alpha_2}{hL}\right) & -\frac{1}{h}\frac{x}{L} & \alpha_1\left(\frac{1}{2L}-\frac{z\alpha_2}{hL}\right) \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix}, \quad (3)
$$

Many authors[1,14] likewise assume that the strain within the adhesive is constant through the thickness. This assumption requires that

$$\alpha_2 = 0$$

**Stress-strain equations**

Stress and strain within the adhesive are related thus

$$\begin{Bmatrix} \sigma_z \\ \sigma_x \\ \tau_{xz} \end{Bmatrix} = [\sigma] = [D]\{\varepsilon\} \tag{5}$$

where for plane stress (IPLANE = 0)

$$[D] = \frac{E_a}{1-\nu_a^2} \begin{bmatrix} \alpha_5 & \alpha_4 \nu_\alpha & 0 \\ \alpha_4 \nu_a & \alpha_4 & 0 \\ 0 & 0 & \dfrac{1-\nu_a}{2} \end{bmatrix} \tag{6}$$

and for plane strain (IPLANE = 1)

$$[D] = \frac{E_a(1-\nu_a)}{(1+\nu_a)(1-2\nu_a)} \begin{bmatrix} \alpha_5 & \dfrac{\alpha_4 \nu_a}{1-\nu_a} & 0 \\ \dfrac{\alpha_4 \nu_a}{1-\nu_a} & \alpha_4 & 0 \\ 0 & 0 & \dfrac{1-2\nu_a}{2(1-\nu_a)} \end{bmatrix} \tag{7}$$

where

$E_a$ = the modulus of elasticity of the adhesive,
$\nu_a$ = Poisson's ratio for the adhesive,
$\alpha_4 = 1$ for the complete stress-strain equations
$= 0$ for incomplete stress-strain equations as discussed below,
$\alpha_5 = 1$ for the consistent stress-strain equations
= other value for inconsistent stress-strain equations as discussed below.

Goland and Reissner[1] (referred to as GR) and Ojalvo and Eidinoff[12] (referred to as OE) assumed the following stress-strain relationship for the adhesive

$$\sigma_z = E_a \varepsilon_z \quad \text{(for GR and OE)}$$

To model this assumption (whicn is a violation of the stress–strain equations) one should take

$$\alpha_5 = 1 - \nu_a^2 \quad \text{(for plane stress and GR and OE)}$$

or

$$\alpha_5 = \frac{(1+\nu_a)(1-2\nu_a)}{(1-\nu_a)} \quad \text{(for plane strain and GR and OE)}$$

and

$$\alpha_4 = 0 \quad \text{(for GR and OE)}$$

**Element derivation**

The stiffness matrix $[k_a]$ relates the element nodal forces to the element nodal displacements thus

$$\begin{Bmatrix} X_1 \\ Z_1 \\ X_2 \\ Z_2 \\ X_3 \\ Z_3 \\ X_4 \\ Z_4 \end{Bmatrix} = [k_a] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix} \tag{8}$$

or

$$\{P\} = [k_a]\{\delta\} \tag{9}$$

The matrix $[k_a]$ is given by[31]

$$[k_a] = \int_V [B]^t[D][B]\, dV \tag{10}$$

For a unit width configuration, Eq. (10) becomes

$$[k_a] = \int_{x=0}^{x=L} \int_{z=-h/2}^{z=h/2} [B(x, z)]^t[D][B(x, z)]\, dx\, dz \tag{11}$$

If the $\xi$, $\eta$ reference system is employed, Eq. (11) then becomes

$$[k_a] = \frac{hL}{4} \int_{\xi=-1}^{\xi=1} \int_{\eta=-1}^{\eta=1} [B(\xi, \eta)]^t[D][B(\xi, \eta)]\, d\xi\, d\eta \tag{12}$$

Integration of Eq. (12) is conveniently done using the Gaussian Quadrature integration formulae.[31] In this investigation, 2-point quadrature was used to integrate in both the $\xi$ and $\eta$ directions.

**Offset nodes**

If the adhesive element is to be used with beam or plate elements which have their nodes along the centroids of the elements, the nodes of the adhesive elements will have to be offset. This offset can be accomplished with a rigid body transformation of forces and displacements. Figure 3 shows the adhesive element with offset nodes. A bar over a symbol indicates "at the offset nodes." From

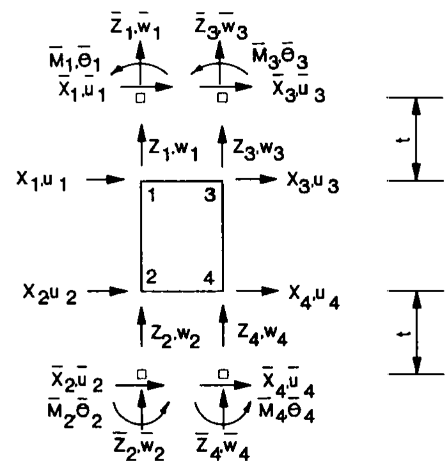

FIGURE 3 Offset nodes.

equilibrium of the rigid body connecting the nodes of the adhesive element to the offset nodes (see Figure 4),

$$\begin{Bmatrix} \bar{X}_i \\ \bar{Z}_i \\ \bar{M}_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ t & 0 \end{bmatrix} \begin{Bmatrix} X_i \\ Z_i \end{Bmatrix} \tag{13}$$

and

$$\begin{Bmatrix} \bar{X}_j \\ \bar{Z}_j \\ \bar{M}_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -t & 0 \end{bmatrix} \begin{Bmatrix} X_j \\ Z_j \end{Bmatrix} \tag{14}$$

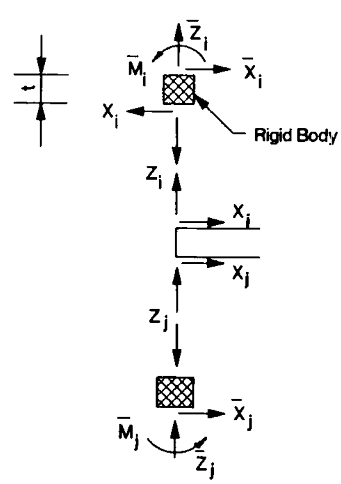

FIGURE 4 Forces on rigid bodies.

$$\{\bar{P}\} = \begin{Bmatrix} \bar{X}_1 \\ \bar{Z}_1 \\ \bar{M}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_4 \\ \bar{Z}_4 \\ \bar{M}_4 \end{Bmatrix}, \qquad \{\bar{\delta}\} = \begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_4 \\ \bar{w}_4 \\ \bar{\theta}_4 \end{Bmatrix} \tag{15}$$

Thus from eqns (13) and (14)

$$\begin{Bmatrix} \bar{X}_1 \\ \bar{Z}_1 \\ \bar{M}_1 \\ \bar{X}_2 \\ \bar{Z}_2 \\ \bar{M}_2 \\ \bar{X}_3 \\ \bar{Z}_3 \\ \bar{M}_3 \\ \bar{X}_4 \\ \bar{Z}_4 \\ \bar{M}_4 \end{Bmatrix} = \left[\begin{array}{cc:cc:cc:cc} 1 & 0 & & & & & & \\ 0 & 1 & & & & & & \\ t & 0 & & & & & & \\ \hdashline & & 1 & 0 & & & & \\ & & 0 & 1 & & & & \\ & & -t & 0 & & & & \\ \hdashline & & & & 1 & 0 & & \\ & & & & 0 & 1 & & \\ & & & & t & 0 & & \\ \hdashline & & & & & & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & -t & 0 \end{array}\right] \begin{Bmatrix} X_1 \\ Z_1 \\ X_2 \\ Z_2 \\ X_3 \\ Z_3 \\ X_4 \\ Z_4 \end{Bmatrix} \tag{16}$$

or

$$\{\bar{P}\} = [T]\{P\} \tag{17}$$

In those theories which consider that the stress in the adhesive is constant through its thickness, the deformation characteristics of the adhesive are defined by the quantities $E_a/h$ and $G_a/h$ and not by the parameters $E_a$, $G_a$, and $h$ themselves. Thus, it is possible to treat the adhesive as having zero thickness with properties defined by $E_a/h$ and $G_a/h$. Goland and Reissner[1] and Delale and Erdogan[14] treat the adhesive in this way. With the finite element approach for this case, the stresses from the adhesive are assumed to be transferred to the adherends at a distance of $t_b/2 + h/2$ from the centroid of the adherends as shown in Figure 5a. This approach was used in Ref. 28. This situation is referred to in this paper as the zero thickness of adhesive assumption. For the case of zero adhesive thickness then

$$t = \frac{t_b}{2} + \frac{h}{2}, \qquad \text{IFIN} = 0 \tag{18}$$

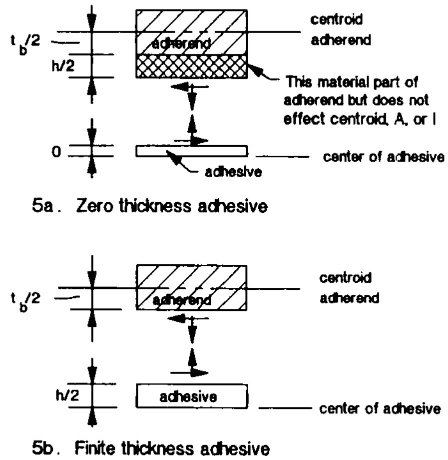


FIGURE 5 Treatment of adhesive.



Figure 5b shows the case where stresses from the adhesive are transferred to the adherend at the actual edges of the adherends. Ojalvo and Eidinoff[12] treated the adhesive in this way and a finite element model of their assumptions[29] does likewise. This situation is referred to as the finite thickness of adhesive assumption. For this case

$$t = \frac{t_b}{2}, \qquad \text{IFIN} = 1 \tag{19}$$

The stiffness matrix $[\bar{k}_a]$ relating $\{\bar{P}\}$ to $\{\bar{\delta}\}$ is given by[31]

$$[\bar{k}_a] = [T]^t[k_a][T] \tag{20}$$

## TWO NODE ADHESIVE ELEMENT

Figure 6 shows a 2-node adhesive element. The displacements within the element are assumed to be of the form[29]

$$u = \frac{(u_1 + u_2)}{2} + \frac{z}{h}(u_1 - u_2)$$

$$w = \frac{(w_1 + w_2)}{2} + \frac{z}{h}(w_1 - w_2) \tag{21}$$

$$w' = \frac{\partial w}{\partial x} = \frac{(w_1' + w_2')}{2} + \frac{z}{h}(w_1' - w_2')$$

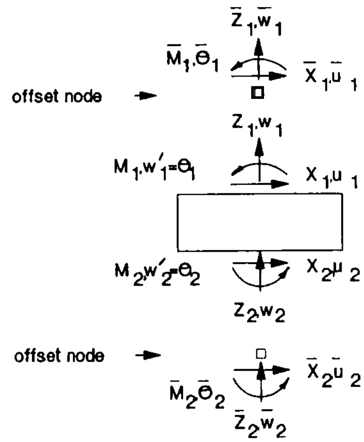


FIGURE 6 Two node adhesive element.

strain within the element is given by

$$\begin{Bmatrix} \varepsilon_z \\ \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 0 & b_1 & 0 & 0 & -b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & b_2 & -b_1 & 0 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ w_1' \\ u_2 \\ w_2 \\ w_2' \end{Bmatrix} \tag{22}$$

where

$$b_1 = \frac{1}{h}$$
$$b_2 = \alpha_1\left[1/2 + \frac{\alpha_2 z}{h}\right] \tag{23}$$
$$b_3 = \alpha_1\left[1/2 - \frac{\alpha_2 z}{h}\right]$$

and where $\alpha_1$ and $\alpha_2$ are defined in the 4-node adhesive element derivation. As the strain does not vary in the $x$ direction with this element, the integrations of Eq. (10) can be easily performed in closed form to give

$$[k_a] = \begin{bmatrix} c_1 & 0 & c_3 & -c_1 & 0 & c_3 \\ 0 & c_2 & 0 & 0 & -c_2 & 0 \\ c_3 & 0 & c_4 & -c_3 & 0 & c_5 \\ -c_1 & 0 & -c_3 & c_1 & 0 & -c_3 \\ 0 & -c_2 & 0 & 0 & c_2 & 0 \\ c_3 & 0 & c_5 & -c_3 & 0 & c_4 \end{bmatrix} \tag{24}$$

where for a unit width configuration

$$
\begin{aligned}
c_1 &= \frac{GL}{h} \\
c_2 &= \alpha_5 D_{11} L/h \\
c_3 &= c_1 \alpha_1 h/2 \\
c_4 &= c_1 \frac{h^2}{4}\left[\alpha_1^2 + \frac{\alpha_1^2\alpha_2^2}{3}\right] \\
c_5 &= c_1 \frac{h^2}{4}\left[\alpha_1^2 - \frac{\alpha_1^2\alpha_2^2}{3}\right]
\end{aligned} \tag{25}
$$

where $D_{11}$ is the 1, 1 term of the matrix $[D]$ in Eqs. (6) or (7).

If one models the assumptions of Goland and Reissner[1] or Ojalvo and Eidinoff[12] by taking

$$\alpha_4 = 0$$

$$\alpha_5 = (1 - \nu_a^2) \quad \text{(for plane stress)}$$

$$\alpha_5 = \frac{(1+\nu_a)(1-2\nu_a)}{(1-\nu_a)} \quad \text{(for plane strain)}$$

and if $\alpha_2$ of Ref. 29 is replaced by $\alpha_1\alpha_2$ to account for differences in usage of $\alpha_2$ between Ref. 29 and this paper, the stiffness matrix of Eq. (24) is found to correspond to that of Ref. 29.

The stiffness matrix $[\bar{k}_a]$ is then given by Eq. (20) where

$$
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
t & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -t & 0 & 1
\end{bmatrix} \tag{26}
$$

## THE ADHEREND ELEMENT

The adherends are modeled with beam-type elements. The element idealization is shown in Figure 7. Bending and axial deformation is considered and shear deformation may or may not be considered depending on a control parameter.

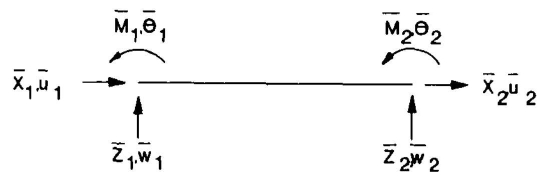


FIGURE 7 Bean element.

Let

$$\{\bar{P}_b\}=\begin{Bmatrix}\bar{X}_1\\ \bar{Z}_1\\ \bar{M}_1\\ \bar{X}_2\\ \bar{Z}_2\\ \bar{M}_2\end{Bmatrix},\qquad \{\bar{\delta}_b\}=\begin{Bmatrix}\bar{u}_1\\ \bar{w}_1\\ \bar{\theta}\\ \bar{u}_2\\ \bar{w}_2\\ \bar{\theta}_2\end{Bmatrix} \tag{27}$$

Then[32]

$$\{\bar{P}_b\}=[\bar{k}_b]\{\bar{\delta}_b\} \tag{28}$$

where

$$[\bar{k}_b]=\begin{bmatrix}\bar{c}_1 & 0 & 0 & -\bar{c}_1 & 0 & 0\\ 0 & \bar{c}_2 & \bar{c}_3 & 0 & -\bar{c}_2 & \bar{c}_3\\ 0 & \bar{c}_3 & \bar{c}_4 & 0 & -\bar{c}_3 & \bar{c}_5\\ -\bar{c}_1 & 0 & 0 & \bar{c}_1 & 0 & 0\\ 0 & -\bar{c}_2 & -\bar{c}_3 & 0 & \bar{c}_2 & -\bar{c}_3\\ 0 & \bar{c}_3 & \bar{c}_5 & 0 & -\bar{c}_3 & \bar{c}_4\end{bmatrix} \tag{29}$$

and where

$E$ = the modulus of elasticity of the adherends,
$\nu$ = Poisson's ratio of the adherends,
$I$ = moment of inertia of the adherends under plane stress conditions,
$A$ = area of the adherends under plane stress conditions,
$I^*$ = $I$ for plane stress
= $I/(1-\nu^2)$ for plane strain,
$A^*$ = $A$ for plane stress
= $\alpha_6 A/(1-\nu^2)$ for plane strain,
$\alpha_6$ = 1 for a consistent plane strain assumption for the adherends
= other value for inconsistent assumptions as discussed below,
$A_e$ = the effective area in shear,[32]
$\alpha_3$ = 1 if shear deformation of the adherends is considered
= 0 if shear deformation neglected,

$$\begin{aligned} \bar{c}_1 &= \frac{A^*E}{L} \\ \bar{c}_2 &= \frac{12EI^*}{L^3(1+\phi)} \\ \bar{c}_3 &= \frac{6EI^*}{L^2(1+\phi)} \\ \bar{c}_4 &= \frac{(4+\phi)EI^*}{L(1+\phi)} \\ \bar{c}_5 &= \frac{(2-\phi)EI^*}{L(1+\phi)} \\ \phi &= \frac{12EI^*\alpha_3}{GA_eL^3} \end{aligned} \tag{30}$$

For rectangular shaped adherends of unit width

$$\begin{aligned} A &= t_b \\ I &= 1/12(t_b^3) \\ A_e &= 5/6(t_b) \end{aligned} \tag{31}$$

where $t_b$ is the thickness of the adherend.

Goland and Reissner[1] took the adherends to be in plane strain ($I^* = I/(1-\nu^2)$) when considering bending but used, inconsistently, plane stress when considering axial forces ($A^* = A$). To model Goland and Reissner's assumption

$$\alpha_6 = 1 - \nu^2 \quad \text{(plane strain and GR)}$$



### Review of terms

Table I lists the various control parameters in this paper and the significance of those parameters.

## THEOREY OF DELALE AND ERDOGAN[14]

Delale and Erdogan assume that stresses are constant through the thickness of the adhesive. Thus

$$\alpha_2 = 0 \tag{32}$$

in Eq. (3). They also assume an incomplete shear strain-displacement equation, *i.e.*

$$\gamma_{xz} = \frac{\partial u}{\partial z} \tag{33}$$

Thus

$$\alpha_1 = 0 \tag{34}$$

TABLE I
Control parameters

| Parameter | Significance |
|---|---|
| IPLANE | = 1 for plane strain<br>= 0 for plane stress |
| IFIN | = 0 for zero thickness adhesive assumption<br>= 1 for finite thickness adhesive assumption |
| $\alpha_1$ | = 0 for incomplete shear–strain displacement assumption for the adhesive<br>= 1 for complete shear–strain displacement assumption for the adhesive |
| $\alpha_2$ | = 0 if adhesive strain does not vary with $z$<br>= 1 if adhesive strain does vary with $z$ |
| $\alpha_3$ | = 0 if shear deformation of the adherends is not considered,<br>= 1 if shear deformation is considered |
| $\alpha_4$ | = 0 if certain terms in the stress–strain equations for the adhesive are neglected<br>= 1 if those terms are not neglected |
| $\alpha_5$ | = 1 if the consistent stress–strain equations for the adhesive are considered<br>= other value if inconsistent equations considered |
| $\alpha_6$ | = 1 if consistent plane strain assumption for the adherend used<br>= other value in inconsistent assumption used |



in Eq. (3). They consider that a $\sigma_x$ stress can exist in the adhesive and thus use the complete and consistent stress–strain equations. Thus

$$\alpha_4 = 1 \quad \alpha_5 = 1 \tag{35}$$

in Eq. (7). These authors consider the case of plane strain, thus

$$\text{IPLANE} = 1 \tag{36}$$

Because of the plane strain assumption, for the adherends

$$A^* = A/(1 - \nu^2) \quad I^* = I/(1 - \nu^2) \tag{37}$$

Because of the consistent use of plane strain for the adherends by these authors, there is is no correction on the $A^*$ term. Thus

$$\alpha_6 = 1 \tag{38}$$

These authors also used the zero thickness of adhesive assumption. Thus

$$\text{IFIN} = 0 \tag{39}$$

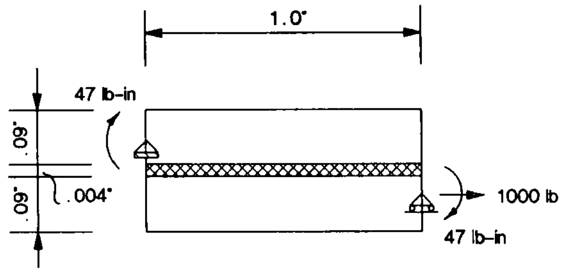


FIGURE 8 Membrane loading.

They considered both bending and shear deformation for the adherends. Thus

$$\alpha_3 = 1 \tag{40}$$

in Eq. (24).

## EXAMPLE 1

This first example is presented to show that finite element analyses using the adhesive element developed can duplicate results from analytical lap joint theories. The theory of Delale and Erdogan[14] has removed many of the inconsistencies of earlier theories and is thus considered in this example. Delale and Erdogan examined three types of loading on a simple lap joint. The loading which they labeled Membrane Loading is shown in Figure 8 and is the loading considered in this example. Particulars of the problem which they considered are

$$E = 10^7 \text{ psi} \qquad \nu = 0.3$$

$$E_a = 5.797 \times 10^5 \text{ psi} \qquad \nu_a = 0.3027$$

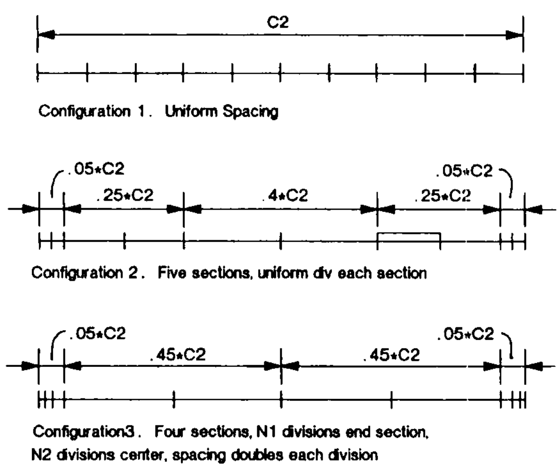


FIGURE 9 Divisions of the adherends.

Three finite element idealizations were made of the configuration of Figure 8. The differences between the idealizations is the manner in which the adherends (and thus the adhesive) were subdivided into elements. The three methods of subdivision are shown in Figure 9. Control parameters were set to those considered by Delale and Erdogan[14] thus

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_3 &= 1 \\ \alpha_4 &= 1 \\ \alpha_5 &= 1 \\ \alpha_6 &= 1 \\ \text{IFIN} &= 0 \\ \text{IPLANE} &= 1 \end{aligned} \tag{42}$$

Figures 10 and 11 show the results of numerous finite element analyses using the two-node adhesive element of this paper. Figure 10 shows the maximum adhesive shear stress *versus* the number of divisions of the top adherend (and thus the bottom adherend) and Figure 11 shows the maximum adhesive normal stress *versus* the number of divisions of the top adherend. One can see that the shear stress in the adhesive and the normal stress in the adhesive converge to the results of those authors. Configurations 2 and 3 performed better than configuration 1 because of the larger number of beam and adhesive elements located near the ends of the configuration, the region where the adhesive stress is changing rapidly.

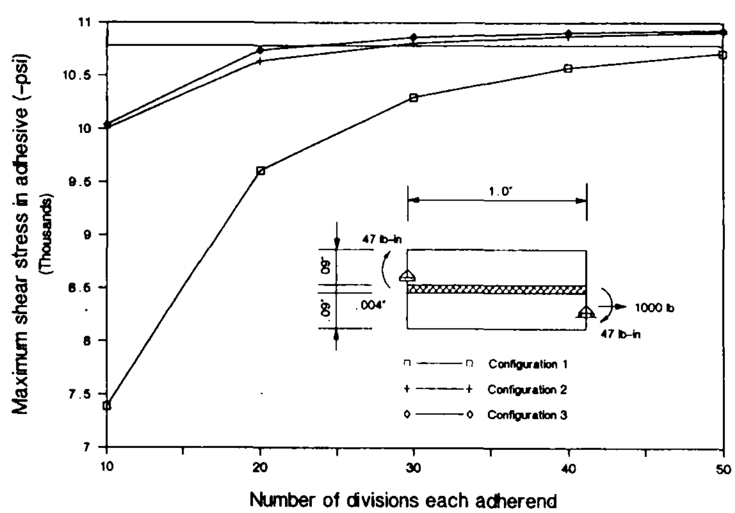


FIGURE 10 Maximum adhesive shear stress using 2-node adhesive element.

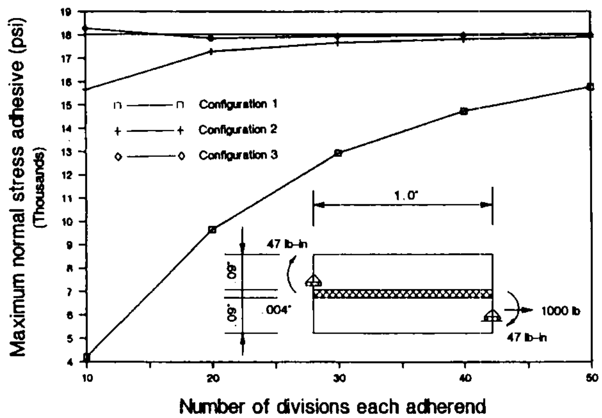


FIGURE 11 Maximum adhesive normal stress using 2-node adhesive element.

Figures 12 and 13 show results using the 4-node adhesive element of this paper with the same set of control parameters as for the 2-node element. One can see that results for both maximum adhesive shear stress and maximum adhesive normal stress also converge to those of Delale and Erodgan. As before, configurations 2 and 3 gave better results than configuration 1.

## EXAMPLE 2

This example studies the effect of the various control parameters on the configuration defined in Example 1. In this study, the idealization of Configuration 2 in Figure 9 was used with both the top and bottom adherends having 50 divisions. In Example 1, control parameters were set in Eq. (42) to model the

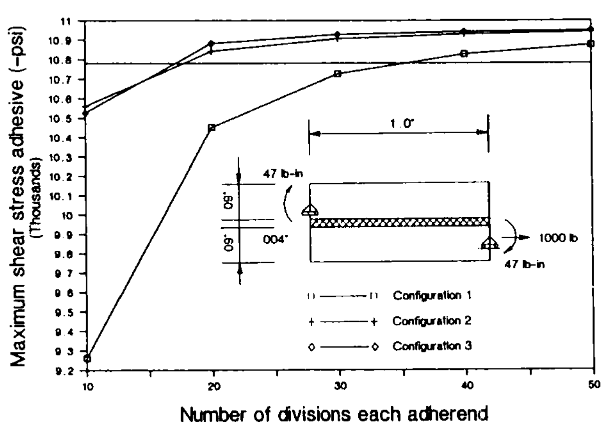


FIGURE 12 Maximum adhesive shear stress using 4-node adhesive element.

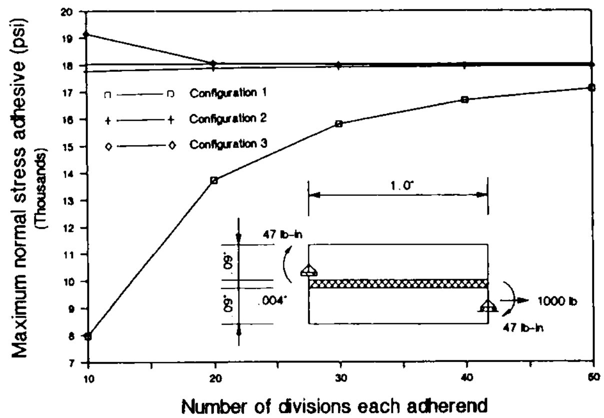


FIGURE 13 Maximum adhesive normal stress using 4-node adhesive element.



assumptions of Delale and Erdogan. In this example, results using the assumptions of Delale and Erdogan were taken as the standard against which results using other assumptions were compared. Table II gives the maximum shear and normal stress in the adhesive for a number of sets of control parameters. In each, one parameter at a time is varied from the standard to examine the effects of the various assumptions. The physical situation which these control parameters model is next discussed.

*Case 1.* Delale and Erdogan assumed plane strain for the adhesive and adherends. Here plane stress for both is assumed.

*Case 2.* Delale and Erdogan and Goland and Reissner made the assumption of zero thickness adherend (see previous section for discussion). Here the finite thickness assumption is made.

TABLE II
Assumption cases considered

| Case | IPLANE | IFIN | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\tau_{max}$ (psi) | $\sigma_{max}$ (psi) |
|---|---|---|---|---|---|---|---|---|---|---|
| Std | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | −10,949 | 17,982 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | −11,478 | 16,989 |
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | −10,914 | 17,880 |
| 3 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | −10,971 | 18,088 |
| 4 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | −10,937 | 17,972 |
| 5 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | −10,958 | 18,079 |
| 6 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | −10,949 | 17,992 |
| 7 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | −10,949 | 18,053 |
| 8 | 1 | 0 | 0 | 0 | 1 | 1 | 0.737 | 1 | −10,949 | 15,443 |
| 9 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0.910 | −11,075 | 17,979 |
| 10 | 1 | 0 | 0 | 0 | 1 | 0 | 0.737 | 1 | −10,949 | 15,503 |
| 11 | 1 | 0 | 0 | 0 | 0 | 0 | 0.737 | 0.910 | −11,075 | 15,510 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −10,929 | 17,977 |

*Case 3.* The complete strain-displacement equations require that $\alpha_1 = 1$ and $\alpha_2 = 1$ in Eq. (3). In this case, $\alpha_1 = 1$ with $\alpha_2 = 0$. With this assumption, strain does not vary in the $z$-direction.

*Case 4.* In this case $\alpha_2 = 1$ but $\alpha_1 = 0$. Here, strain is allowed to vary in the $z$-direction but an incomplete shear–strain displacement equation is assumed (*i.e.* $\gamma_{xz} = \partial u/\partial z$).

*Case 5.* Here $\alpha_1 = \alpha_2 = 1$. Thus, the complete strain displacement equations are being used.

*Case 6.* Delale and Erdogan considered shear deformation of the adherend and thus took $\alpha_3 = 1$. Here, $\alpha_3 = 0$. Thus, shear deformation of the adherends is neglected.

*Case 7.* Delale and Erdogan used the complete and considtent strain-displacement equations for the adhesive. Goland and Reissner, on the other hand, used $\alpha_4 = 0$ in Eq. (6). One of the ramifications of using $\alpha_4 = 0$ is that one thus obtains $\sigma_x = 0$.

*Case 8.* One of the most salient features of the theory of Goland and Reissner is that they assumed $\sigma_z = E\varepsilon_z$. If plane strain is being assumed for the adhesive, then one must take $\alpha_5 = (1 + \nu_a)(1 - 2\nu_a)/(1 - \nu_a)$ to accomplish their assumption (See Eq. (7)). In this example $\nu_a = 0.3027$ giving $\alpha_5 = 0.7372$.

*Case 9.* Goland and Reissner took the adherends to be in plane strain when considering bending but used, inconsistently, plane stress when considering axial force. To model Goland and Reissner's assumptions, one must take $\alpha_6 = 1 - \nu^2$ which with $\nu = 0.3$ yields $\alpha_6 = 0.91$.

*Case 10.* Here, by taking $\alpha_6 = 1$, the effect on Goland and Reissner's results of using a consistent plane strain assumption is examined.

*Case 11.* Here the original Goland and Reissner assumptions are modeled.

*Case 12.* In case 12, the complete strain-displacement equations are used and consistent with this assumption, the finite adhesive assumption is made. Here also the complete and consistent stress strain equations are used.

One can see in Table II that maximum adhesive stress is insensitive to most of the assumptions made about the behavior of the adhesive and adherends. The exception is that the assumption made by Goland and Reissner that $\sigma_z = E\varepsilon_z$ (inconsistent stress–strain equations) affects the normal stress by approximately 15%.

Case 12 uses the consistent stress–strain equations and the complete strain-displacement equations. Using the assumptions of case 12, as the finite element grid was refined, results converged to those of Table II. Thus, it is not just the incomplete strain-displacement and/or incomplete stress–strain equations which yields stress convergence with mesh refinement. Here, the single row of adhesive elements coupled with the displacement incompatibility between the adherend and adhesive elements permits convergence. If one used more than one row of adhesive elements and compatible plane stress or plane strain elements for both the adherends and adhesive, the stress singularity at the joint's edges would prevent convergence of the finite element solution with some refinement.

## CONCLUSION

A two-node and a four-node adhesive element are presented for modeling the adhesive in a bonded configuration. The elements can be used in conjunction with standard 2D plane stress or plane strain elements to model the adherends or can have their nodes offset to allow them to be used with beam or plate type elements which have their nodes at their centroids. Using these elements together with standard elements to model the adherends allows the power of the finite element method to be used to analyze complex geometries. Because the adhesive elements are derived using incomplete strain-displacement equations, the stress singularities which would otherwise occur at the connection edges at the bi-material interfaces are removed and convergence of adhesive stress is thus obtained with mesh refinement.

The elements are quite general and can model, with the selection of values of control parameters, the assumptions of numerous authors. The assumptions of Delale and Erdogan[14] are discussed in detail and it is shown that the adhesive element with appropriate control parameters gives results which converge to those of that reference. The effect of various assumptions on the maximum shear and normal stress in the adhesive was investigated. Results were found to be insensitive to assumptions made with the exception that the inconsistent stress–strain assumption of Goland and Reissner[1] affected results by approximately 15%.

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